THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2060B Mathematical Analysis II
Homework 1 Suggested Solutions

1. (6.1.6 of [BS11]) Let $n \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^n$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$. For which values of n is f' continuous at 0? For which values of n is f' differentiable at 0?

Solution. We know that the derivative f' is given by

$$f'(x) = \begin{cases} nx^{n-1}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

We see clearly that $\lim\limits_{x \to 0^-} f'(x) = 0$, so in order for f' to be continuous at 0, we need $\lim\limits_{x \to 0^+} f'(x) = 0$ as well. We have

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} nx^{n-1} = \begin{cases} 1, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

So we conclude that f' is continuous at 0 for $n \geq 2$.

For differentiability of f' at 0, we know that continuity is a necessary condition, so we already have that $n \geq 2$ at least. By definition of the derivative,

$$\lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0^-} \frac{0 - 0}{x} = 0,$$

so for differentiability of f' at 0, we again expect $\lim\limits_{x \to 0^+} \dfrac{f'(x) - f'(0)}{x} = 0$ as well. We have

$$\lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0^+} \frac{nx^{n-1}}{x} = \lim_{x \to 0^+} nx^{n-2} = \begin{cases} 1, & n = 2 \\ 0, & n \geq 3 \end{cases}$$

So we conclude that f' is differentiable at 0 for $n \geq 3$. ◀

2. (6.1.9 of [BS11]) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an **even function** [that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an **odd function** [that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$]. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function.

Solution. Since f is even, $f(-x + h) = f(-(x - h)) = f(x - h)$, and so

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} \\ &= -\lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} = -f'(x) \end{aligned}$$

as required.

Likewise, since g is odd, $g(-x+h) = -g(x-h)$, and we have

$$\begin{aligned} g'(-x) &= \lim_{h\to 0} \frac{g(-x+h)-g(-x)}{h} = \lim_{h\to 0} \frac{-g(x-h)+g(x)}{h} \\ &= \lim_{h\to 0} \frac{g(x-h)-g(x)}{-h} = g'(x) \end{aligned}$$

as required. ◀

3. (6.1.15 of [BS11]) Given that the restriction of the cosine function cos to $I := [0,\pi]$ is strictly decreasing and that $\cos 0 = 1$, $\cos\pi = -1$, let $J := [-1,1]$, and let $\mathrm{Arccos} : J \to \mathbb{R}$ be the function inverse to the restriction of cos to I. Show that Arccos is differentiable on $(-1,1)$ and $D\mathrm{Arccos}y = (-1)/(1-y^2)^{1/2}$ for $y \in (-1,1)$. Show that Arccos is not differentiable at -1 and 1.

   Solution. By Theorem 6.1.8 of [BS11], Arccos is differentiable on $(-1,1)$ with derivative at given by
   $$D\mathrm{Arccos}y = \frac{1}{\cos' x} = -\frac{1}{\sin x}$$
   where x is such that $y = \cos x$. Given this relationship, we know that $1 = y^2 + \sin^2 x \Rightarrow \sin x = \sqrt{1-y^2}$ and so we have that
   $$D\mathrm{Arccos}y = -\frac{1}{\sqrt{1-y^2}}$$
   as required.

   From the formula found for $D\mathrm{Arccos}y$, we readily see that the right hand side is not well-defined at $y = \pm 1$, and so Arccos is not differentiable at ± 1. ◀

4. (6.2.5 of [BS11]) Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Prove that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$. [*Hint*: Show that $f(x) := x^{1/n} - (x-1)^{1/n}$ is decreasing for $x \geq 1$, and evaluate f at 1 and a/b.]

   Solution. We first note that if $a > 0$, then the function $g(x) = x^{-a}$ is decreasing for $x \geq 0$. We can see this by observing that for $x \geq 0$, $g'(x) = -ax^{a-1} \leq 0$ and so we can conclude that g is decreasing for $x \geq 0$ by Theorem 6.2.7 of [BS11].

   Following the hint, we want to show that the function $f(x) = x^{1/n} - (x-1)^{1/n}$ is decreasing for $x \geq 1$. Taking derivative, we have that
   $$f'(x) = \frac{1}{n}x^{1/n-1} - \frac{1}{n}(x-1)^{1/n-1}.$$
   Observe that for $n \geq 2$, the exponent $1/n - 1 < 0$ and so by above the function $g(x) = \frac{1}{n}x^{1/n-1}$ is decreasing for $x \geq 0$. Hence, for $x \geq 1$, we have that
   $$\frac{1}{n}(x-1)^{1/n-1} < \frac{1}{n}x^{1/n-1},$$

that is, that $f'(x) < 0$ for $x \geq 1$. Hence, evaluating at 1 and $a/b > 1$ (since $a > b > 0$), we have that

$$f\left(\frac{a}{b}\right) < f(1) \Longrightarrow \left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} < 1^{\frac{1}{n}} - (1-1)^{\frac{1}{n}} = 1.$$

Multiplying by $b^{\frac{1}{n}}$ on both sides, we obtain

$$a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} < b^{\frac{1}{n}} \Longleftrightarrow a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}$$

as required. ◀

5. (6.2.10 of [BS11]) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) := 0$. Show that $g'(0) = 1$, but in every neighborhood of 0 the derivative $g'(x)$ takes on both positive and negative values. Thus g is not monotonic in any neighborhood of 0.

Solution. The function g is given by

$$g(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

By definition, we find that

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} \frac{x + 2x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} 1 + 2x \sin\left(\frac{1}{x}\right) = 1$$

using the inequality $|\sin x| \leq 1$ for all $x \in \mathbb{R}$.

For $x \neq 0$, by the Chain rule and properties of the derivative (Theorem 6.1.3 of [BS11]), we find that

$$g'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right).$$

We find two sequences (x_n), (y_n) such that $x_n, y_n \to 0$ as $n \to +\infty$ such that $g'(x_n) < 0$ and $g'(y_n) > 0$ for all $n \in \mathbb{N}$. Let

$$x_n := \frac{1}{2n\pi}, \quad y_n := \frac{2}{(4n+1)\pi},$$

then we see that both $x_n, y_n \to 0$ as $n \to +\infty$ and we have that

$$g'(x_n) = 1 + 4\left(\frac{1}{2n\pi}\right)\sin(2n\pi) - 2\cos(2n\pi) = 1 - 2 = -1 < 0$$

while

$$\begin{aligned} g'(y_n) &= 1 + 4\left(\frac{2}{(4n+1)\pi}\right)\sin\left(\frac{(4n+1)\pi}{2}\right) - 2\cos\left(\frac{(4n+1)\pi}{2}\right) \\ &= 1 + \frac{8}{(4n+1)\pi} > 0 \end{aligned}$$

for all $n \in \mathbb{N}$ as required. ◀

6. (6.2.13 of [BS11]) Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I.

   Solution. Let $x_1 < x_2 \in I$. Then by the Mean Value Theorem, we have that there is a $c \in (x_1, x_2)$ such that

   $$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

   Since $x_2 - x_1 > 0$ and $f'(c) > 0$, we then see that

   $$f(x_2) - f(x_1) > 0 \iff f(x_1) < f(x_2)$$

   as required. ◀

References

[BS11] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis, Fourth Edition*. Fourth. University of Illinois, Urbana-Champaign: John Wiley & Sons, Inc., 2011. ISBN: 978-0-471-43331-6.